Some theoretical results concerning the displacement of a viscous oil by a hot fluid in a porous medium

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Two simultaneous first-order non-linear equations are derived to give a 'high-flow-rate' model for the displacement of oil by hot water in a porous medium. The solution of these equations is analysed by the method of characteristics and it is shown that in problems for which thermal capacity dependence on temperature is neglected, the solution will have the properties of a simple wave. The simple wave behaviour gives a rapid method for solving practical systems. When temperature dependence is included in the thermal capacities, it is found that a simultaneous shock in temperature and saturation develops, but the solution will usually approximate quantitatively the simple wave result. Decoupling the equations by using an average saturation in the heat transport equation gives results in reasonable agreement with the coupled case.

Introduction

Present techniques for the recovery of a highly viscous crude from a petroleum reservoir, using the solution-gas-drive or natural-water-drive process provided by nature, yield remarkably small percentage recoveries of the oil in place (sometimes as small as a few per cent). The forces provided by nature are often supplemented by the injection of water in suitably located wells, but apart from maintaining the pressure within an oil field, the injected water is at best an inefficient driving mechanism. When heat is introduced into the reservoir, the increase in temperature reduces the viscosity of the crude, and the efficiency of the displacement by a driving fluid is improved. Many of the present attempts to utilize heat in the reservoir are focused on the use of water, both as a convector of heat, and as the displacing fluid. Two attractive methods for heating the water are to flood it through a portion of the reservoir in which an *in situ* combustion of oil has occurred, or alternatively in which an atomic bomb has been exploded. The economic attractiveness of such a scheme will depend upon a balance between the extra costs associated with introducing heat into the reservoir and the value of the extra recovered oil.

At normal reservoir conditions an approximate description of the behaviour of the water-drive system may be obtained from the solution of the Buckley-Leverett frontal advance equation (see Buckley & Leverett 1942)

$$\frac{q}{\phi}\frac{df_w}{dS}\frac{\partial S}{\partial x}+\frac{\partial S}{\partial t}=0; \tag{1}$$

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f_w is the fractional flow of water, related to laboratory-determined permeability functions, $k_w(S)$ and $k_o(S)$, by

$$f_w(S) = \left[1 \pm \left(\frac{\rho_w - \rho_o}{\mu_o q}\right) g k_o \sin\alpha\right] \left(1 + \frac{k_o}{k_w}\frac{\mu_w}{\mu_o}\right)^{-1}. \tag{2}$$

Here S is the water saturation, ρ the density, ϕ the porosity, μ the viscosity, q the total fluid flux (assumed constant in this analysis), g the acceleration due to gravity and α the tilt angle of the one-dimensional system. The subscripts w and o refer to water and oil respectively. When $\alpha = 0$, the fractional flow function has the S-shaped character indicated in figure 1.

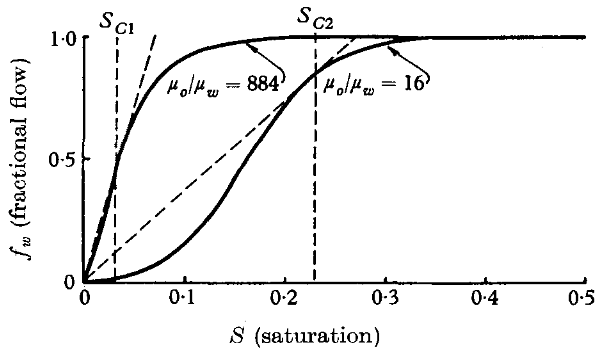

FIGURE 1. Fractional flow curves.

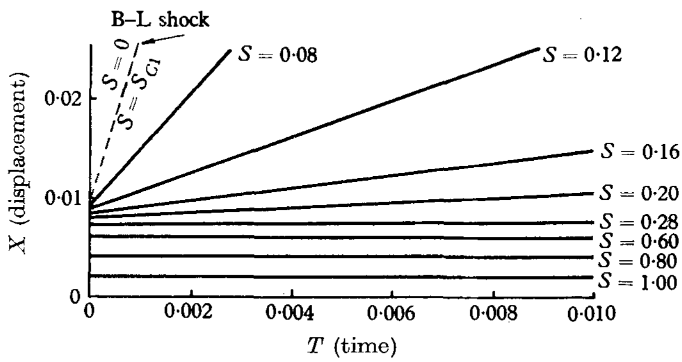

FIGURE 2. Characteristic diagram for simple water flood.

Equation (1) is a non-linear first-order equation which may be derived from Darcy's Law and a material balance condition. It is easily solved by the method of characteristics,* the characteristic direction being given by

$$\frac{dx}{dt} = \frac{q}{\phi}\frac{df_w}{dS}, \tag{3}$$

while the appropriate characteristic relation is

$$dS/dt = 0. \tag{4}$$

* The characteristic method for solving equation (1) was not used by Buckley & Leverett (1942), but has been briefly discussed in this context by Scheidegger (1957).

Since S is constant along a characteristic, the characteristics are straight lines. The nature of the f_w function leads to the intersection of characteristics; the intersection may be interpreted physically as the formation of a shock. Methods similar to those used in the theory of hyperbolic flow (as described by Courant & Friedrichs 1948 or Von Mises 1958) may be used to map the characteristics and shock path numerically in a step-by-step process. A typical characteristic diagram and saturation profile is illustrated in figures 2 and 3. Consideration

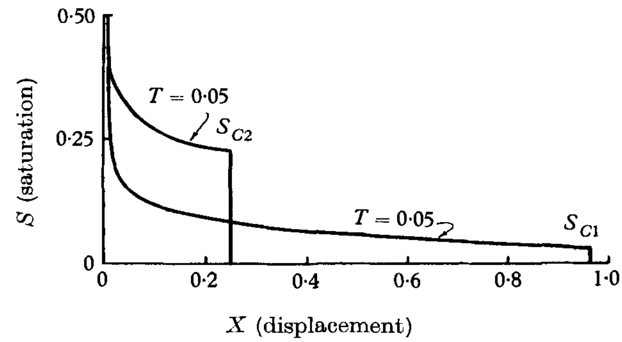

FIGURE 3. Saturation profiles for two viscosity ratios.

of the S-shaped character of the f_w function indicates that an equilibrium shock strength will rapidly develop, this being determined by the saturation value at which the tangent from the origin touches the f_w curve (see figure 1). The height of the saturation discontinuity determines the efficiency of the displacement mechanism. When the oil is highly viscous, the discontinuity is very weak and the water moves ahead rapidly, displacing little oil.

Simplified equations for hot-water displacement

A linear system having water injected at constant rate and constant temperature is to be studied. It is assumed that the rock material in any small element of the system will be in thermal equilibrium with the two flowing phases, which are also assumed to be in thermal equilibrium. (There is a resonable quantity of experimental evidence to support this assumption.) The mathematical model will be further idealized to assume that the fluid-filled porous medium is a sufficiently good insulator for fluid flux to become the principal heat transport mechanism. These assumptions, combined with the fluid flow approximations, are needed to give a hyperbolic system. In practice, of course, the fluid flow equations should include the effects of forces associated with interfacial tensions, while the heat equation should include three-dimensional thermal diffusivity terms. The system would then be parabolic and shock behaviour would be eliminated. However, much valuable information and sufficiently accurate solutions may usually be obtained by solving the simpler hyperbolic model. The hyperbolic system may be regarded as the 'high-flow-rate' model. Inclusion of the temperature (θ) dependence in $f_w(S, \theta)$ gives a modified fluid flow equation

$$q\frac{\partial f_w}{\partial S}\frac{\partial S}{\partial x}+\phi\frac{\partial S}{\partial t}+q\frac{\partial f_w}{\partial \theta}\frac{\partial \theta}{\partial x} = 0. \tag{5}$$

Conservation of thermal energy in the system demands that

$$q\sigma_w \frac{\partial}{\partial x}(f_w \theta) + q\sigma_o \frac{\partial}{\partial x}\{(1-f_w)\theta\} + \phi\left[\sigma_w \frac{\partial}{\partial t}(S\theta) + \sigma_o \frac{\partial}{\partial t}\{(1-S)\theta\}\right] + \sigma_R \frac{\partial \theta}{\partial t} + \text{small viscous term} = 0. \tag{6}$$

The σ's are defined as the thermal capacities per unit volume of water, oil, and porous rock. Putting the equations into dimensionless form and simplifying (6), we obtain

$$\frac{\partial f_w(S,\Pi)}{\partial S}\frac{\partial S}{\partial X} + \frac{\partial S}{\partial T} + \frac{\partial f_w(S,\Pi)}{\partial \Pi}\frac{\partial \Pi}{\partial X} = 0, \tag{7}$$

$$A\frac{\partial \Pi}{\partial X} + B\frac{\partial \Pi}{\partial T} = 0, \tag{8}$$

where $X = x/L$ (fractional length), $T = qt/L\phi$ (total pore volume flow),

$$\Pi = (\theta - \theta_{\min})/(\theta_{\max} - \theta_{\min}) \quad \text{(reduced temperature)},$$

and $$A(S,\Pi) = f_w\left(1 - \frac{\sigma_o}{\sigma_w}\right) + \frac{\sigma_o}{\sigma_w}, \quad B(S,\Pi) = S\left(1 - \frac{\sigma_o}{\sigma_w}\right) + \frac{\sigma_o}{\sigma_w} + \frac{\sigma_R}{\phi\sigma_w}.$$

Equations (7) and (8) have been derived with the assumption that fluid density and heat capacity are not functions of temperature. In practice, densities are slowly varying functions of temperature, and heat capacities show a somewhat larger temperature dependence. Derivation of (8) on the basis of conservation of enthalpy gives the same result, but the σ's are now functions of Π. When density is treated as a function of the dependent variable, the system loses its hyperbolic property; consequently this variation has not been included in the analysis. The temperature dependence is best chosen on the basis of constant mass rather than constant volume, since mass flow rate will tend to be constant in regions of thermal gradient. It will be seen that the introduction of this functional dependence considerably influences the nature of the characteristic diagram, but has little effect on the final quantitative results.

Shock conditions

The analysis of non-linear hyperbolic flow often requires the concept of shock formation. In the introduction we noted that the solution of the simple water-flood problem led to the formation of the so-called Buckley–Leverett shock. Scheidegger (1957) has shown that application of a material balance requirement at the shock front gives for the saturation shock velocity

$$\frac{d\xi_s}{dT} = \frac{f_{w+} - f_{w-}}{S_+ - S_-}, \tag{9}$$

where + 's refer to values just ahead of the shock and − 's to values just behind the shock. When a discontinuity in temperature occurs, a coincident discontinuity in saturation must also take place. Conservation of enthalpy at a simultaneous saturation and temperature-shock will be satisfied if

$$\phi \frac{dx_\theta}{dt}\left[S\int_{\theta_0}^{\theta}\sigma_w d\theta + (1-S)\int_{\theta_0}^{\theta}\sigma_o d\theta + \int_{\theta_0}^{\theta}\frac{\sigma_R}{\phi}d\theta\right]$$
$$= q\left[f_w\int_{\theta_0}^{\theta}\sigma_w d\theta + (1-f_w)\int_{\theta_0}^{\theta}\sigma_o d\theta\right]. \qquad (10)$$

The square bracket, [ ], implies a difference between + and − values. In dimensionless form

$$\frac{d\xi_\theta}{dT} = \frac{[f_w(H_w-H_o)]+[H_o]}{[S(H_w-H_o)]+[H_o]+[H_R]}, \qquad (11)$$

where $[H_{w,o}] = \int_{\theta_-}^{\theta_+}\sigma_{w,o}d\theta/H, \quad [H_R] = \int_{\theta_-}^{\theta_+}\sigma_R d\theta/\phi H.$

H is a normalization constant which may conveniently be taken as the enthalpy difference for water between $\theta_{\min}$ and $\theta_{\max}$.

We make the physically reasonable hypothesis that the saturation and temperature discontinuities associated with the temperature shock have the same velocity. Thus from (9),

$$\frac{d\xi_\theta}{dT} = \frac{[f_w]}{[S]}. \qquad (9a)$$

By combining (11) and (9a) using an algebraic identity of the kind

$$\frac{a}{b} = \frac{c}{d} = \frac{a-\gamma c}{b-\gamma d},$$

we obtain

$$\frac{d\xi_\theta}{dT} = \frac{f_{w+}\{[H_w]-[H_o]\}+[H_o]}{S_+\{[H_w]-[H_o]\}+[H_o]+[H_R]}. \qquad (12)$$

Equations (9a) and (12) may be thought of as the Rankine–Hugoniot relations for a hot water flood. Equation (12) indicates that a knowledge of S_+, Π_+ and Π_- are adequate to determine $d\xi_\theta/dT$. The dependent variable S_- can be determined readily from (9a). Similarly S_-, Π_+ and Π_- will determine $d\xi_\theta/dT$ and S_+. When the thermal capacities are constant, (12) may be simplified to

$$\frac{d\xi_\theta}{dT} = \left\{f_{w+}\left(1-\frac{\sigma_o}{\sigma_w}\right)+\frac{\sigma_o}{\sigma_w}\right\}\Big/\left\{S_+\left(1-\frac{\sigma_o}{\sigma_w}\right)+\frac{\sigma_o}{\sigma_w}+\frac{\sigma_R}{\phi\sigma_w}\right\}. \qquad (13)$$

Equation (13) depends only on Π_+ and S_+.

Simple-wave solution of constant-thermal-capacity problem

The use of the method of characteristics in the solution of simultaneous first-order equations has been discussed by Courant & Friedrichs (1948). Using their analysis, we see that (7) and (8) are homogeneous and reducible and thus may be linearized by means of a hodograph transformation to the form

$$\frac{\partial f_w}{\partial S}\frac{\partial T}{\partial \Pi} - \frac{\partial f_w}{\partial \Pi}\frac{\partial T}{\partial S} - \frac{\partial X}{\partial \Pi} = 0, \qquad (14)$$

$$-A\frac{\partial T}{\partial S} + B\frac{\partial X}{\partial S} = 0. \qquad (15)$$

The transformation is valid provided the Jacobian

$$j = \frac{\partial(S,\Pi)}{\partial(X,T)} \neq 0, \tag{16}$$

and similarly the solution to the image system is a solution of the primary system, provided

$$J = \frac{\partial(X,T)}{\partial(S,\Pi)} \neq 0. \tag{17}$$

For problems in which $j = 0$, the transformation is not helpful and there is a region of the primary solution called a 'simple wave'. The simple-wave region maps into a single characteristic in the hodograph plane, say a Γ_+ characteristic. Along this characteristic $J = 0$. We analyse the primary system. The characteristic directions for (7) and (8) can be obtained from

$$\begin{vmatrix} \frac{\partial f_w}{\partial S} - \frac{dX}{dT} & \frac{\partial f_w}{\partial \Pi} \\ 0 & \frac{A}{B} - \frac{dX}{dT} \end{vmatrix} = 0;$$

that is

$$\frac{dX}{dT} = \frac{\partial f_w(S,\Pi)}{\partial S}, \tag{18}$$

or

$$\frac{dX}{dT} = \frac{A(S,\Pi)}{B(S,\Pi)}. \tag{19}$$

Along these directions the characteristic relations are given by

$$\begin{vmatrix} \frac{dS}{dT} & \frac{\partial f_w}{\partial \Pi} \\ \frac{d\Pi}{dT} & \frac{A}{B} - \frac{dX}{dT} \end{vmatrix} = 0,$$

so that, when $dX/dT = A/B \quad (\partial f_w/\partial\Pi \neq 0)$,

$$d\Pi/dT = 0 \quad \text{or} \quad \Pi = \text{const.} \tag{20}$$

When $dX/dT = \partial f_w/\partial S$,

$$\frac{dS}{dT} = \frac{\partial f_w}{\partial \Pi}\frac{d\Pi}{dT} \bigg/ \left(\frac{A}{B} - \frac{\partial f_w}{\partial S}\right), \tag{21a}$$

but

$$\frac{d\Pi}{dT} = \frac{\partial \Pi}{\partial X}\frac{dX}{dT} + \frac{\partial \Pi}{\partial T},$$

so that using this result together with (8) we obtain the characteristic relation

$$\frac{dS}{dT} = -\frac{\partial f_w}{\partial \Pi}\frac{\partial \Pi}{\partial X}. \tag{21}$$

It is of interest to note that (21) is the same form as would have been obtained had we treated (7) separately, regarding $-(\partial f_w/\partial\Pi)\,\partial\Pi/\partial X$ as a source term.

Equations (18), (19), (20) and (21) are ordinary differential equations which may be integrated numerically in a step-by-step process. The intersection of characteristics can be interpreted in a shock theory using (9) and (12). It is convenient to assume as initial condition a small linear penetration of S and Π, the remaining boundary conditions being specified by

$$S = 1{\cdot}0, \quad \Pi = 1{\cdot}0 \quad \text{at} \quad X = 0. \tag{22}$$

The functional relations determining the characteristic directions, f_w and A/B, as determined from typical oil and reservoir properties, are illustrated in figures 4 and 5. The resulting characteristic diagram is illustrated in figure 6. Typical saturation and temperature profiles are indicated in figure 7.

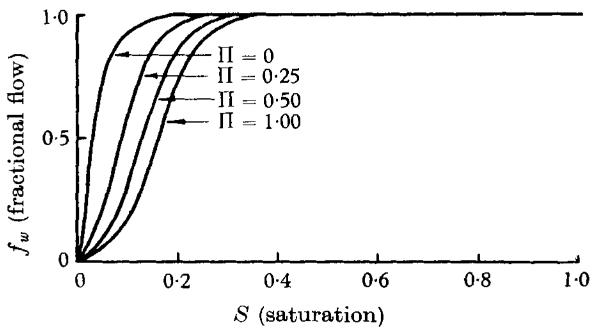

FIGURE 4. Fractional flow function.

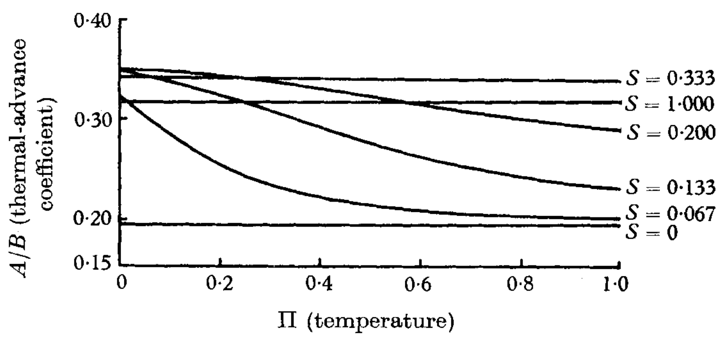

FIGURE 5. Thermal-advance coefficient for constant-thermal-capacity problem.

After a brief initial phase, there are four principal regions in the characteristic diagram. In region I, saturations are moving ahead of the temperature bank in a reservoir at ambient temperature, $\Pi = 0$. The saturation shock strength and velocity are identical to that of a simple Buckley–Leverett solution. Region II is a saturation plateau of constant saturation at ambient temperature. In region III, the S-characteristics curve up across straight-line Π-characteristics. Each Π-characteristic has a constant temperature associated with it (as required by (20)) and also a constant saturation value. In region IV, the solution is identical to part of a simple water-flood with the temperature at its maximum value, $\Pi = 1$.

The fact that region II is a region of constant state, while region III has straight-line Π-characteristics, is consistent with a simple-wave phenomenon. Consequently, we expect $j = 0$ in region III, and expect there to be a Γ_+-characteristic for which $J = 0$. By using the condition $j = 0$ and referring to (7) and (8), we find that $j = 0$ is equivalent to

$$\frac{A}{B}\frac{\partial S}{\partial X}+\frac{\partial S}{\partial T} = 0, \tag{23}$$

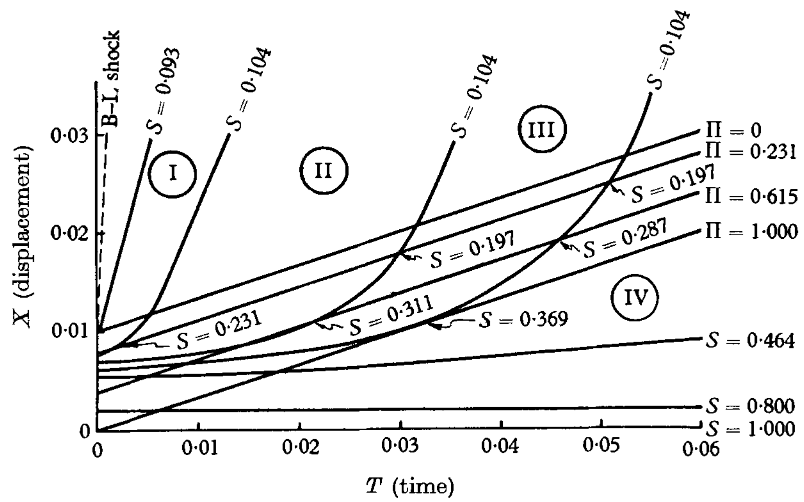

FIGURE 6. Characteristic diagram for constant-thermal-capacity problem.

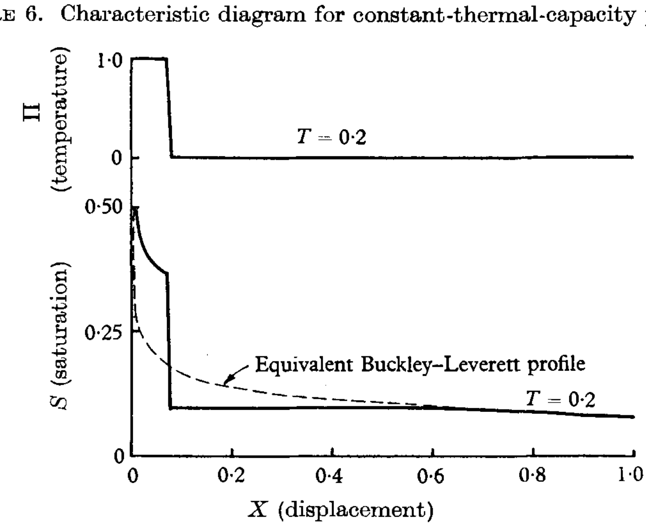

FIGURE 7. Typical saturation and temperature profiles for constant-thermal-capacity problem.

which requires $dS = 0$ along the directions $dX/dT = A/B$. Thus, saturation is constant along a Π-characteristic in a simple-wave region. Using (23) to eliminate $\partial S/\partial T$ in (7), we obtain

$$\left(\frac{\partial S}{\partial \Pi}\right)_T = -\frac{\partial f_w}{\partial \Pi}\bigg/\left(\frac{\partial f_w}{\partial S}-\frac{A}{B}\right). \tag{24}$$

Equation (24) is an important inner relation for the simple-wave region. Given an initial distribution for Π in the simple wave, (24) determines the corresponding constant S-profile in the wave. Furthermore, the characteristic directions for the hodograph equations, (14) and (15), are given by

$$\frac{dS}{d\Pi} = -\frac{\partial f_w}{\partial \Pi}\Big/\left(\frac{\partial f_w}{\partial S}-\frac{A}{B}\right), \tag{25}$$

and when $J = 0$ it can be shown that $dT = 0$ along this direction; thus (24) also gives the Γ_+-characteristic which maps the simple-wave region in the hodograph plane.

Finally, we show that the Π-characteristics are parallel in the simple-wave region:

$$\frac{\partial}{\partial X}\left(\frac{A}{B}\right) = \frac{1}{B}\left(\frac{\partial A}{\partial \Pi}\frac{\partial \Pi}{\partial X}+\frac{\partial A}{\partial S}\frac{\partial S}{\partial X}\right)-\frac{A}{B^2}\left(\frac{\partial B}{\partial \Pi}\frac{\partial \Pi}{\partial X}+\frac{\partial B}{\partial S}\frac{\partial S}{\partial X}\right)$$

and, using (24), together with the definitions of A and B in (8), we find for the simple-wave region that

$$\frac{\partial}{\partial X}\left(\frac{A}{B}\right) = 0, \tag{26}$$

provided σ_o, σ_w and σ_R are not functions of Π.

The simple-wave nature of the solution for constant heat capacity allows a convenient technique for obtaining solutions to practical problems. Inasmuch as an arbitrary linear penetration was chosen for the initial Π-distribution, and the fact that this remained almost unchanged after the brief initial phase, we can arbitrarily assume a sharp linear distribution for Π in the simple wave, $\Pi = CX + VT$. The saturation distribution in the wave would then be obtained from (24). Therefore, when the initial Π-distribution is indefinitely sharp, the corresponding S-distribution will also be indefinitely sharp. In this case there is a saturation front propagated in association with a temperature front. The velocity, V, and S_- for the front are determined from

$$V = \frac{A(S_-, 1)}{B(S_-, 1)} = \frac{\partial f_w(S_-, 1)}{\partial S}, \tag{27}$$

while S_+ for the front is determined from

$$V = A(S_+, 0)/B(S_+, 0). \tag{28}$$

For $S > S_-$, the solution is the Buckley–Leverett type corresponding to $S > S_-$ and $\Pi = 1$. For $S < S_+$, the solution is the Buckley–Leverett type for $S < S_+$ and $\Pi = 0$. This region and the foot of the temperature front are joined by a plateau of constant saturation with value S_+. The functional form of

$$A(S, \Pi)/B(S, \Pi) \quad \text{and} \quad f_w(S, \Pi) \quad \text{for} \quad 0 < \Pi < 1$$

would not be needed in this type of solution.

Solution of variable-thermal-capacity problem

When the temperature dependence is introduced into the thermal capacities, there is no longer a simple relationship between the function A/B and f_w. In consequence, (26) is not fulfilled, and a simple-wave solution with parallel Π-characteristics is not possible. The modified A/B function is illustrated in figure 8. Construction of the characteristic diagram for this case (figures 9 and 10) gives converging Π-characteristics which are *nearly* straight lines. The convergence of the Π-characteristics leads to the gradual formation of a temperature shock moving with an associated saturation shock. With the particular choice of an initial linear penetration, the shape of the temperature-shock path is concave downward. For early T-values, the S-characteristics move into the shock from above, thus determining the value of S_+. The shock velocity and S_- for this situation are obtained from (9a) and (12). There are two possible values of S_- which fulfill the mathematical requirements, the preferred value being selected by physical considerations. The shock must be backward-facing because

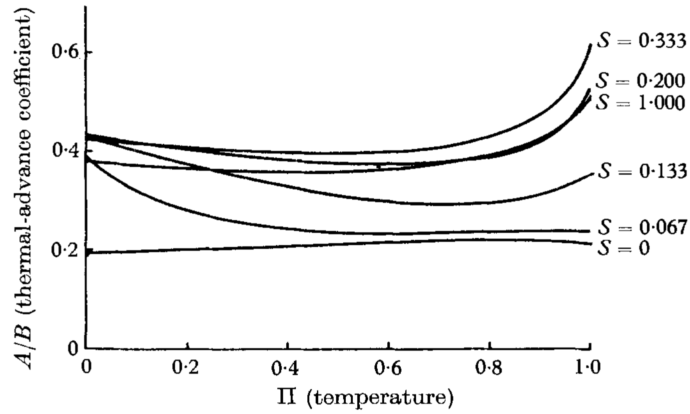

FIGURE 8. Thermal-advance coefficient for variable-thermal-capacity problem.

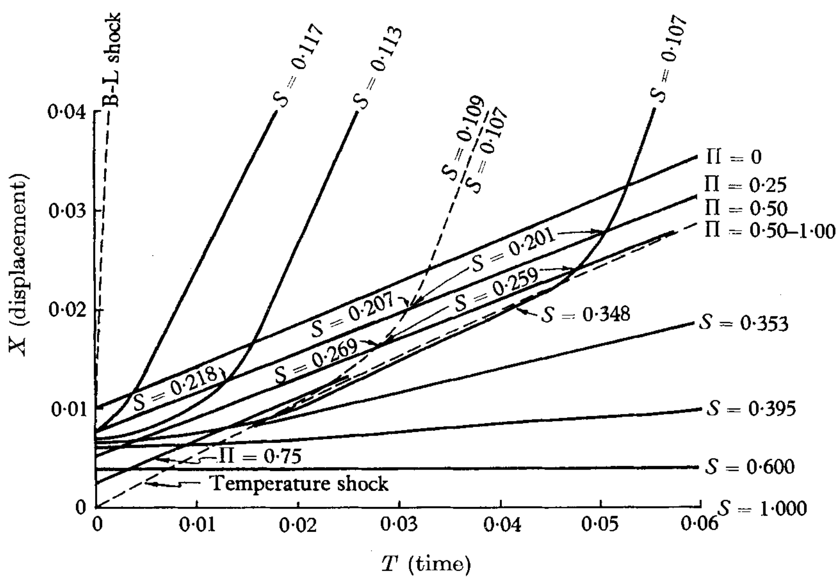

FIGURE 9. Characteristic diagram for variable-thermal-capacity problem.

S_--characteristics are moving more slowly than the shock velocity. We select the S_- value which tends to S_+ as the temperature strength tends to zero. At a later time the shock velocity decreases to a stage where the calculated S_--characteristic becomes parallel to the shock path, and thereafter previous S_--characteristics re-enter the shock path. At this instant the physical and mathematical requirements demand that the shock break into two parts. The entry of S_--characteristics from below causes the formation of a forward-facing s ock with S_+-characteristics moving ahead at higher velocities than the shock velocity. These faster-moving S_+-characteristics lead to the formation of a small backward-facing saturation shock moving ahead of the main temperature shock. These phenomena are illustrated in figure 11 and 12.

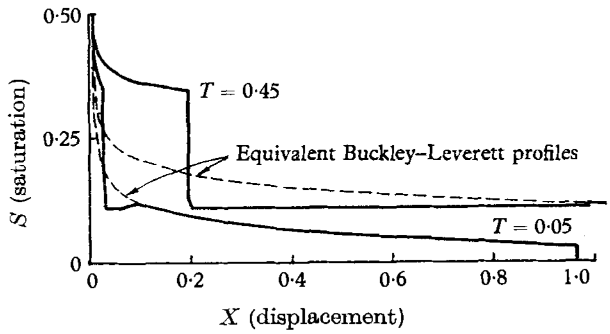

FIGURE 10. Typical saturation curves for variable-thermal-capacity problem.

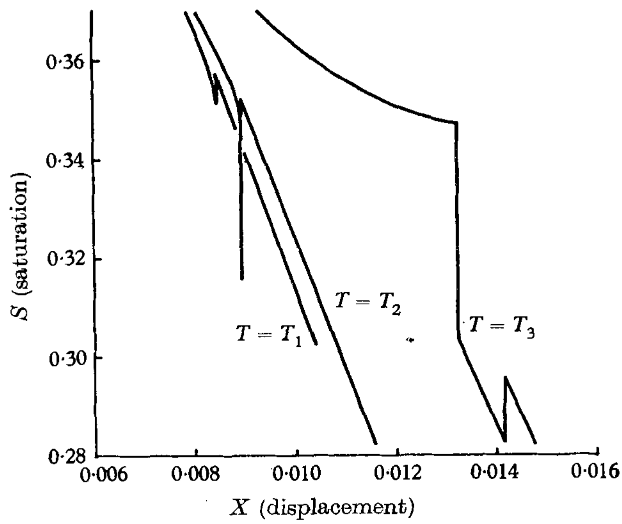

FIGURE 11. Partial saturation profiles illustrating formation of double shock.

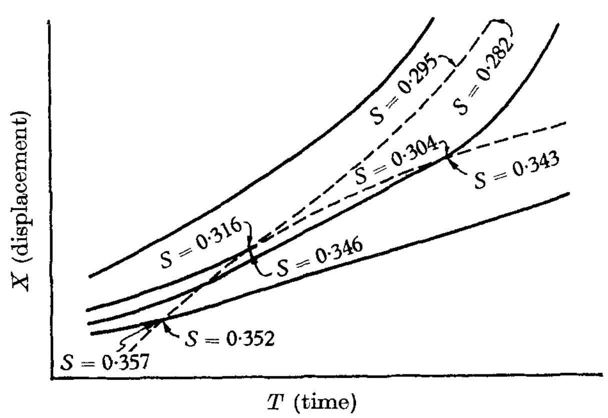

FIGURE 12. Schematic diagram of characteristics in region of double-shock formation.

After a brief period, the leading shock advances into the $\Pi = 0$ region and has a slowly changing plateau of saturation behind it in this region. The plateau corresponds to the region of constant state in the previous simple-wave problem. Eventually the Π-characteristics converge to form a stabilized temperature shock, at which time S in the plateau remains fixed, and the solution henceforth behaves similarly to the simple-wave solution. At all times the quantitative behaviour of the variable thermal capacity solution is close to that of the simple-wave solution, even though the nature of the characteristic diagrams is different. It is concluded that the only significant difference between the two solutions is that thermal-capacity variation will cause an initial temperature distribution to steepen gradually into a shock, whereas the simple-wave nature of the constant-capacity case leaves an initial distribution almost unaltered. This conclusion depends on the functional dependence chosen for the thermal capacities.

Solution of decoupled equations

Much of the present effort in predicting the performance of thermal drive processes is dependent upon the assumption that temperature profiles may be calculated neglecting saturation variation and the multiphase flow behaviour of the system. This approximate temperature behaviour is then used to calculate the saturation behaviour. In the high-flow-rate model, this concept would be expressed by using an average saturation value for A/B in (8), thus reducing it to an equation in one unknown. The solution of the decoupled equation can easily be determined by the method of characteristics. The temperature term, $-(\partial f_w/\partial \Pi)\,\partial \Pi/\partial X$, is then treated as a space-and-time-varying source in the solution of (7). The validity of this approach was tested for the variable-thermal-capacity problem. The average saturation was chosen as the value associated with the tangent to the $f_w(S, 0{\cdot}5)$-curve. The solution of the decoupled problem was found to differ by not more than 1 or 2 % from the coupled solution for all calculated temperatures and saturations. However, solution of a decoupled system would not have given the simple-wave solution associated with the constant-thermal-capacity problem.

On the basis of this result it would seem reasonable to continue the decoupling philosophy in introducing extra physics into the equations describing a thermal-flood process. One simple improvement in this respect would be to express a heat-loss term as proportional to the difference between instantaneous and ambient temperature. The resulting modification in (8) would cause the Π-characteristics to become curved.

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